Lower bound for the number of nodes of cubature formulae on the unit ball

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Abstract

A lower bound for positive cubature formulae on the unit ball is given. For the Chebyshev weight function on the ball in $\mathbb{R}^2$, the new bound shows that a positive cubature formula of degree $s$ with all nodes inside the ball will need at least

$$N_s \geq 0.13622 s^2 \left( 1 + \mathcal{O}(s^{-1}) \right)$$

number of nodes, in comparison with the classical lower bound of

$$N_s \geq 0.125 s^2 \left( 1 + \mathcal{O}(s^{-1}) \right).$$

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1. Introduction

Let $W$ be a nonnegative weight function defined on $\mathbb{R}^d$ and denote by $\mathcal{L}(f) = \int_{\mathbb{R}^d} f(x) W(x) \, dx$. A cubature formula of degree $s$ with respect to $W$ is a linear functional

$$\mathcal{L}_s(f) = \sum_{k=1}^{N_s} \lambda_k f(x_k), \quad \lambda_k \in \mathbb{R}, \quad x_k \in \mathbb{R}^d,$$

such that $\mathcal{L}_s(f) = \mathcal{L}(f)$ for all polynomials $f$ of degree at most $s$, and there exists at least one polynomial $f^{*}$ of degree exactly $s + 1$ for which $\mathcal{L}_s(f^{*}) \neq \mathcal{L}(f^{*})$. If all cubature weights $\lambda_k$ are positive, we call $\mathcal{L}_s$ a positive cubature. The purpose of this paper is to provide a lower bound for $N_s$ for the weight function

$$W_\mu(x) = w_\mu \left( 1 - ||x||^2 \right)^{\mu-1/2}, \quad w_\mu = \frac{\Gamma(\mu + \frac{d+1}{2})}{\pi^{d/2} \Gamma(\mu + \frac{1}{2})}.$$
on the unit ball $B^d = \{ x : \| x \| \leq 1 \}$, where $\| x \|^2 = x_1^2 + \cdots + x_d^2$ and $\mu \geq 0$. In the case $\mu = 0$, $W_0$ is called the Chebyshev weight function on the unit ball. The weight function $W_\mu$ is normalized so that it has unit integral over $B^d$. As a consequence, $\sum_{k=1}^{N_s} \lambda_k = 1$ for every cubature formula.

Let $\Pi^d_n$ denote the space of polynomials of degree $n$ in $d$ variables. It is known that the number of nodes $N_s$ of any cubature formula of degree $s$ satisfies

$$N_s \geq \dim \Pi^d_{\lfloor s/2 \rfloor} + \left( \left\lfloor \frac{s}{2} \right\rfloor + d \right) \left( \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right),$$

where $\lfloor x \rfloor$ stands for the largest integer less than or equal to $x$. Furthermore, for many weight functions, including $W_\mu$ and all other centrally symmetric weight functions, $N_s$ satisfies Möller’s lower bound if $s$ is an odd integer. For $s = 2m - 1$ and $d = 2$, this bound states

$$N_{2m-1} \geq \dim \Pi^2_{m-1} + \left( \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right)\left( \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right).$$

Bound (1.1) is classical (cf. [8]). Möller’s bound appeared in [7]; it shows that bound (1.1) is not sharp for $W_\mu$ and $s$ being odd. Furthermore, bound (1.2) is not sharp for $W_\mu$ if $m$ is an odd integer [3,10], which means a lower bound of $\frac{m(m+1)}{2} + \left\lfloor \frac{m}{2} \right\rfloor + 1$ for $N_{2m-1}$ in the case of $W_\mu$ and $m$ odd. For cubature formula of even degree, bound (1.1) is the only known lower bound.

In the next section we give a new lower bound for $N_s$ that holds for both $s$ being even and odd and positive cubature formulae. The method of deriving the bound can be traced back to the work of Delsarte et al. [4] for spherical designs, which has been extended and used by many authors for cubature formulae on the unit sphere; see the book of [2] and the references therein. However, as far as we know, this method has not been used for cubature formulae on any other domains. On the other hand, there is a close relation between cubature formulae on the sphere and those on the unit ball (cf. [11]). This relation suggests the extension of the method to the ball.

The method depends on a choice of an extreme function. One such function was constructed by Yudin for $\mu$ being an half integer and used in [13] to get a lower bound for the spherical designs. We extend the definition to all $\mu$ with an elementary approach. Using this function, a new lower bound for the Chebyshev weight function is derived which shows that the lower bound (1.1) and (1.2) are not sharp for $s$ large in the case of positive cubature formulae. In fact, for $d = 2$, the new lower bound shows $N_s \geq 0.13622s^2(1 + O(s^{-1}))$ in contrast with $N \geq 0.125s^2(1 + O(s^{-1}))$ of (1.1) and (1.2). In particular, this shows that (1.1) and (1.2) are far from sharp for $s$ large for positive cubature formulae. Numerical computation shows that the new bound is better for $s \geq 31$ (note that 31 is not of the form $2(2k + 1) + 1$). Further study indicates, however, that Yudin’s function appears to yield improved lower bound only for $W_\mu$ with $\mu$ sufficiently small. In particular, it does not seem to give a better bound for the unit weight function.
The paper is organized as follows. In the next section we derive the lower bound in general. Section 3 discusses Yudin’s construction of extreme function. The lower bounds derived using Yudin’s function are discussed in Section 4.

2. Lower bound for positive cubature formulae

Let \( \mathcal{V}_n^d(W_\mu) \) denote the space of orthogonal polynomials of degree exactly \( n \) with respect to \( W_\mu \) on \( B^d \). Making use of the orthogonality, it follows that a cubature formula \( \mathcal{L}_s(f) \) for \( W_\mu \) holds if and only if

\[
\mathcal{L}_s(f) := \sum_{k=1}^{N_s} \lambda_k f(x_k) = 0, \quad f \in \mathcal{V}_n^d(W_\mu), \quad 1 \leqslant n \leqslant s,
\]

and there is a \( f^* \in \mathcal{V}_s^d(W_\mu) \) such that \( \mathcal{L}_s(f^*) \neq 0 \). For a multiindex \( \alpha \in \mathbb{N}_0^d \), let \( |\alpha| = \alpha_1 + \cdots + \alpha_d \). Let \( \{P_\alpha : |\alpha| = n\} \) denote an orthonormal basis of \( \mathcal{V}_n^d(W_\mu) \). Such a basis can be given explicitly in terms of Gegenbauer polynomials (see, for example, [5]). The reproducing kernel of \( \mathcal{V}_n^d \) is denoted by \( P_n(W_\mu; x, y) \) and it can be given in terms of the orthonormal basis \( \{P_\alpha\} \),

\[
P_n(W_\mu; x, y) = \sum_{|\alpha|=n} P_\alpha(x)P_\alpha(y).
\] (2.1)

Let \( C_n^\mu \) denote the Gegenbauer polynomials of degree \( n \), which are orthogonal polynomials with respect to \( w_\mu(t) = (1 - t^2)^{\mu-1/2} \) and normalized so that \( C_n^\mu(1) = \binom{n+2\mu-1}{n} \). Then the reproducing kernel \( P_n(W_\mu; x, y) \) has a compact formula [12]

\[
P_n(W_\mu; x, y) = c_\mu \frac{n + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} \times \int_{-1}^{1} C_n^{\mu + \frac{d-1}{2}}(\langle x, y \rangle + \sqrt{1 - ||x||^2}\sqrt{1 - ||y||^2}t)(1 - t^2)^{\mu-1} \, dt,
\] (2.2)

where \( \langle x, y \rangle \) denotes the usual Euclidean inner product of \( x, y \in \mathbb{R}^d \) and the constant \( c_\mu \) is given by

\[
c_\mu = \left[ \int_{-1}^{1} (1 - t^2)^{\mu-1} \, dt \right]^{-1} = \frac{\Gamma(\mu + 1/2)}{\sqrt{\pi}\Gamma(\mu)}.
\]

For \( \mu = 0 \), formula (2.2) holds under the limit

\[
\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1 - t^2)^{\mu-1} \, dt = \frac{1}{2} [f(1) + f(-1)].
\] (2.3)
We will need to consider the Fourier expansion of a function \( f \) in terms of the Gegenbauer polynomials. For \( f \in L^2(w_\lambda) \), its Gegenbauer expansion is

\[
f(t) = \sum_{n=0}^{\infty} \hat{f}_n^\lambda C_n^\lambda(t), \quad \hat{f}_n^\lambda = h_n^\lambda \int_{-1}^{1} f(t) C_n^\lambda(t)(1 - t^2)^{\lambda - \frac{1}{2}} dt,
\]

in which \( [h_n^\lambda]^{-1} = \int_{-1}^{1} [C_n^\lambda(t)]^2(1 - t^2)^{\lambda - 1/2} dt \). We need one more definition.

**Definition 2.1.** For \( \lambda \geq 0 \) and \( s \geq 1 \), define

\[
\mathcal{F}_s(\lambda) = \{ f : f \in C[-1, 1], \ f(t) \geq 0, \ t \in [-1, 1], \ \text{and} \ \ f(t) = \sum_{k=0}^{\infty} \hat{f}_k^\lambda C_k^\lambda(t) \ \text{with} \ \hat{f}_k^\lambda \leq 0 \ \text{for} \ k > s \}.
\]

Furthermore, for a continuous function \( F \) on \([-1, 1]\), we define

\[
\sigma(F; r) = c_\mu \int_{-1}^{1} F(r + (1 - r)t)(1 - t^2)^{\mu - 1} dt, \quad 0 \leq r \leq 1.
\] (2.4)

**Theorem 2.2.** Let \( F \) be defined on \([-1, 1]\) such that \( F \in \mathcal{F}_s(\mu + (d - 1)/2) \). Then the number of nodes, \( N_s \), of a positive cubature formula with respect to \( W_\mu \) whose nodes are all inside \( B^d \) satisfies

\[
N_s \geq \min_{0 \leq r \leq 1} \sigma(F; r)/\hat{F}_0^\lambda, \quad \lambda = \mu + \frac{d - 1}{2}.
\]

**Proof.** Let \( F \) be given as in the statement. Consider the function \( \Phi(x, y) \) defined by

\[
\Phi(x, y) = c_\mu \int_{-1}^{1} F(\langle x, y \rangle + \sqrt{1 - ||x||^2} \sqrt{1 - ||y||^2} t)(1 - t^2)^{\mu - 1} dt.
\]

In this proof we write \( \hat{F}_n = \hat{F}_n^\lambda \). Expanding \( F \) in terms of the Gegenbauer polynomials \( C_n^\lambda \) and using formula (2.2), we conclude that

\[
\Phi(x, y) = \sum_{n=0}^{\infty} \hat{F}_n \frac{\lambda}{n + \lambda} P_n(W_\mu; x, y).
\]

By the definition of the reproducing kernel (2.1) in terms of an orthonormal basis,

\[
\sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_\mu; x_k, x_l) = \sum_{|x|=n} \left( \sum_{k=1}^{N_s} \lambda_k P_{\lambda}(x_k) \right)^2 \geq 0, \quad n \in \mathbb{N}_0.
\]

Consequently, using the fact that the cubature formula is of degree \( s \) so that \( \sum_{k=1}^{N_s} \lambda_k P_n(W_\mu; x_k, x) = 0 \) for \( 1 \leq n \leq s \) by (2.1), it follows from the assumption \( \hat{F}_n \leq 0 \)
for \( n > s \) that
\[
I := \sum_{k,l=1}^{N_s} \lambda_k \lambda_l \Phi(x_k, x_l) = \sum_{n=0}^{\infty} \hat{F}_n \frac{\lambda}{n+\lambda} \sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_{\mu}; x_k, x_l)
\]
\[
= \hat{F}_0 \sum_{k,l=1}^{N_s} \lambda_k \lambda_l + \sum_{n=s+1}^{\infty} \hat{F}_n \frac{\lambda}{n+\lambda} \sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_{\mu}; x_k, x_l)
\]
\[
\leq \hat{F}_0 \sum_{k,l=1}^{N_s} \lambda_k \lambda_l = \hat{F}_0.
\]
Since \( F(t) \geq 0 \), it follows that \( \Phi(x, y) \geq 0 \). Hence, for the positive cubature formula we have by the Cauchy inequality that
\[
I = \sum_{k,l=1}^{N_s} \lambda_k \lambda_l \Phi(x_k, x_l) \geq \sum_{k=1}^{N_s} \lambda_k^2 \Phi(x_k, x_k)
\]
\[
\geq \min_{0 \leq r \leq 1} \sigma(F, r) \sum_{k=1}^{N_s} \lambda_k^2 \geq \min_{0 \leq r \leq 1} \sigma(F, r) \sum_{k=1}^{N_s} \lambda_k^2
\]
\[
\geq \min_{0 \leq r \leq 1} \sigma(F, r) \left( \sum_{k=1}^{N_s} \lambda_k \right)^2 / N_s = \min_{0 \leq r \leq 1} \sigma(F, r) / N_s,
\]
where in the second inequality we have used the fact that nodes are inside \( B^d \). These two inequalities give the stated estimate on \( N_s \).

If \( F \) is an increasing function, then \( \sigma(F, r) \) is an increasing function on \([0, 1]\). In this case, we have the following corollary:

**Corollary 2.3.** If, in addition, \( F \) is an increasing function on \([-1, 1] \), then
\[
N_s \geq \sigma(F, 0) / \hat{F}_0 = \frac{c_\mu}{c_{\lambda+1/2}} \frac{\int_{-1}^{1} F(t)(1-t^2)^{\mu-1} \, dt}{\int_{-1}^{1} F(t)(1-t^2)^{\lambda-1/2} \, dt}, \quad \lambda = \mu + \frac{d - 1}{2}.
\]

As we mentioned in the introduction, the idea of the proof goes back to [4]. It has been used for spherical designs, which are cubature formulae on the unit sphere with equal weights, the so-called Chebyshev cubature formulae (see [2] and the reference therein).

### 3. Yudin’s function in \( \mathcal{F}(\lambda) \)

Clearly, any polynomial of degree \( s \) that is nonnegative on \([-1, 1]\) is an element of \( \mathcal{F}_s(\lambda) \), but not every choice gives a good lower bound for \( N_s \). In the case of the spherical designs, many authors have discussed choices of proper polynomials in \( \mathcal{F}_s(\lambda) \). See [2] and the reference therein.
We examine a function in $\mathcal{F}_s(\lambda)$ constructed by Yudin [13] for $\lambda = (d - 2)/2$. The definition in [13] is given in terms of integration on the unit sphere and the proof uses properties of spherical harmonics. In the following, we shall give a more elementary proof that works for all $\lambda \geq 0$.

Let $\gamma_s = \gamma_s^2$ be the largest zero of $C_{s-1}^{\lambda+1}(t)$. Define function $f$ and $g$ by

$$f(t) = \begin{cases} C_s^\lambda(t) - C_s^\lambda(\gamma_s), & t \geq \gamma_s \\ 0, & t < \gamma_s \end{cases}$$

and $g(t) = \begin{cases} 1, & t \geq \gamma_s \\ 0, & t < \gamma_s. \end{cases}$

**Proposition 3.1.** Let $f$ and $g$ be defined as in the above. Then the function

$$F(x) = \int_{-1}^1 g(u)c_\lambda \int_{-1}^1 f(ux + t\sqrt{1 - u^2}\sqrt{1 - x^2})(1 - t^2)^{\lambda-1} dt(1 - u^2)^{\lambda+\frac{1}{2}} du$$

is an element of $\mathcal{F}_{s-1}(\lambda)$. In fact, $\hat{F}_k^\lambda = 0$ and for $k \neq s$,

$$\hat{F}_k^\lambda = (h_k^\lambda)^2 \frac{s(s + 2\lambda)}{(k - s)(k + s + 2\lambda)} C_s^\lambda(\gamma_s) \frac{\lambda}{\gamma_k}(\int_{\gamma_s}^1 C_k^\lambda(y)(1 - y^2)^{\lambda-\frac{1}{2}} dy)^2.$$ 

The proof of this proposition depends on the following lemma.

**Lemma 3.2.** For $s \neq k$ and $x \in [-1, 1]$,

$$\int_x^1 C_s^\lambda(y) C_k^\lambda(y)(1 - y^2)^{\lambda-\frac{1}{2}} dy - a_{s,k} C_s^\lambda(x) \int_x^1 C_k^\lambda(y)(1 - y^2)^{\lambda-\frac{1}{2}} dy$$

$$= b_{s,k}(1 - x^2)^{\lambda+\frac{1}{2}} C_k^\lambda(x) \frac{d}{dx} C_s^\lambda(x),$$

where

$$a_{s,k} = \frac{k(k + 2\lambda)}{(k - s)(k + s + 2\lambda)}, \quad b_{s,k} = -\frac{1}{(k - s)(k + s + 2\lambda)}.$$

**Proof.** First, we recall that the Gegenbauer polynomials satisfy a differential equation [9, p. 80]

$$\frac{d}{dx} \left[ (1 - x^2)^{\lambda+\frac{1}{2}} \frac{d}{dx} C_n^\lambda(x) \right] = -n(n + 2\lambda)(1 - x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x). \quad (3.1)$$

Integrating the differential equation (3.1) gives

$$\int_x^1 C_k^\lambda(y)(1 - y^2)^{\lambda-\frac{1}{2}} dy = \frac{1}{k(k + 2\lambda)}(1 - x^2)^{\lambda+\frac{1}{2}} \frac{d}{dx} C_k^\lambda(x), \quad k > 0. \quad (3.2)$$
Using (3.2) the derivative of the left-hand side of the stated equation is
\[
(a_{s,k} - 1) C_s^j(x) C_k^j(x)(1 - x^2)^{j-1/2} - \frac{a_{s,k}}{k(2k + 2\lambda)} \frac{d}{dx} C_s^j(x) \frac{d}{dx} C_k^j(x)(1 - x^2)^{j+1/2}
\]
\[= -b_{s,k}(1 - x^2)^{j-1/2} \left[ s(s + 2\lambda) C_s^j(x) C_k^j(x) - (1 - x^2) \frac{d}{dx} C_s^j(x) \frac{d}{dx} C_k^j(x) \right].
\]

Using (3.1) again, the last expression is easily seen to be the derivative of the right-hand side of the stated equation. □

Proof of Proposition 3.1. Since \(f\) is a continuous function (in fact, \(f\) is differentiable), it is evident that \(F\) is continuous for \(x \in [-1, 1]\). The fact that \(\gamma_s\) is the largest zero of \((d/dx)C_s^j(x) = 2\lambda C_s^{j+1}(x)\) shows that \(f(x) \geq 0\) on \([-1, 1]\), so that \(F(x) \geq 0\). Let \(a_{s,k}\) be as in the previous lemma. Then
\[
a_{s,k} - 1 = \frac{s(s + 2\lambda)}{(k - s)(k + s + 2\lambda)}.
\]
Consequently, the identity in the previous lemma with \(x = \gamma_s\) shows that for \(k \neq s\),
\[
\hat{f}_k = h_k^\lambda \int_{\gamma_s}^1 (C_s^j(y) - C_s^j(\gamma_s)) C_k^j(y)(1 - y^2)^{j-\frac{1}{2}} dy
\]
\[= h_k^\lambda \frac{s(s + 2\lambda)}{(k - s)(k + s + 2\lambda)} C_s^j(\gamma_s) \int_{\gamma_s}^1 C_k^j(y)(1 - y^2)^{j-\frac{1}{2}} dy.
\]
Moreover, using the fact that \(\int_0^1 |C_k^j(t)|(1 - t^2)^{j-1/2} dt = O(k^{2j-1})\) [9, (7.34.1), p. 173] it is easy to see that \(|\hat{f}_k| = O(k^{-j-1})\); hence, using the fact that \(C_k^j(t) = O(k^{-j-1})\) for \(-1 < t < 1\), it follows that the expansion of \(f\) converges uniformly on every compact set inside \([-1, 1]\). The definition of \(g(x)\) shows that the Fourier coefficients of \(g\) is
\[
\hat{g}_k = h_k^\lambda \int_{\gamma_s}^1 C_k^j(y)(1 - y^2)^{j-\frac{1}{2}} dy.
\]
In particular, Eq. (3.2) shows that \(\hat{g}_s = 0\). The Gegenbauer polynomials satisfy a product formula
\[
C_k^j(u)C_k^j(x) = C_k^j(1)C_k \int_{-1}^1 C_k^j(u + t\sqrt{1 - u^2}\sqrt{1 - x^2})(1 - t^2)^{j-1} dt.
\]
Using this formula, we get
\[
F(x) = \int_{-1}^1 g(u) \sum_{k=0}^{\infty} \hat{f}_k \frac{1}{C_k^j(1)} C_k^j(u)C_k^j(x)(1 - u^2)^{j-1/2} du
\]
\[= \sum_{k=0}^{\infty} \hat{f}_k \hat{g}_k \frac{\lambda}{h_k^\lambda(k + \lambda)} C_k^j(x),
\]
since \(h_k^\lambda C_k^j(1) = h_k^\lambda(k + \lambda)/\lambda\) [9, (4.7.3) and (4.7.15)], from which the formula for \(\hat{F}_k\) follows. That \(F \in \mathcal{F}_{s-1}(\lambda)\) follows from the explicit formula of \(\hat{F}_k\) and from the fact that \(C_s^j(\gamma_s) < 0\). □
For $\lambda = (d - 1)/2$, the function $F$ is defined in [13] by

$$F(\langle a, b \rangle) = \int_{S^{d-1}} f(\langle a, x \rangle)g(\langle x, b \rangle)d\omega(x)$$

with the same $f$ and $g$. The proof in [13] uses the properties of the harmonic functions.

The definition in Proposition 3.1 allows us to derive the following properties of $F$, which shows in particular that $F$ is supported on a small interval, since $\gamma_s \to 1$ as $s \to \infty$.

**Proposition 3.3.** Let $F$ be defined as in the previous proposition. Then $F(x) = 0$ if $x \leq 2\gamma_s^2 - 1$ and $F(x)$ is nondecreasing on $[-1, \gamma_s]$.

**Proof.** Let $\phi(t, u, x) = ux + t\sqrt{1 - u^2}\sqrt{1 - x^2}$ and $\phi(u, x) = \phi(1, u, x)$. Since $g(u) = 0$ for $u \leq \gamma_s$ and $g(u) = 1$ for $u \geq \gamma_s$, we can write

$$F(x) = \int_{-1}^1 c_{\lambda} \int_{-1}^1 f(\phi(t, u, x)) (1 - t^2)^{\lambda-1} dt(1 - u^2)^{\lambda-1/2} du.$$  \hspace{1cm} (3.3)

Assume $x \leq \gamma_s \leq u \leq 1$. Then it is easy to verify that $\partial \phi / \partial u < 0$ for this range of $x$ and $u$, so that $\phi(u, x) \leq \phi(\gamma_s, x)$ for $x \leq \gamma_s$. Furthermore, solving the inequality $\phi(\gamma_s, x) \leq \gamma_s$ gives $x \leq 2\gamma_s^2 - 1$. Clearly, $2\gamma_s^2 - 1 \leq \gamma_s$. Therefore, if $x \leq 2\gamma_s^2 - 1$, then $\phi(t, u, x) \leq \phi(\gamma_s, x) \leq \gamma_s$, so that $f(\phi(t, u, x)) = 0$ for $-1 < t < 1$ and, consequently, $F(x) = 0$.

Next we show that $F$ is nondecreasing for $x \in [-1, \gamma_s]$. Since $f''(x) = \frac{d}{dx} C_s^2(x)$ for $x \geq \gamma_s$, it follows that $f'$ is nonnegative on $[-1, 1]$. Taking derivative of $F$ in (3.3) gives

$$F'(x) = \int_{-1}^1 c_{\lambda} \int_{-1}^1 f'(\phi(t, u, x)) \frac{\partial}{\partial x} \phi(t, u, x)(1 - t^2)^{\lambda-1} dt(1 - u^2)^{\lambda-1/2} du.$$ \hspace{1cm} (3.3)

The assumption that $x \leq \gamma_s \leq u$ implies that $\frac{\partial}{\partial x} \phi(t, u, x) = u - tx\sqrt{1 - u^2}\sqrt{1 - x^2} \geq 0$ which shows that $F'(x) \geq 0$ and $F$ is nondecreasing. \hspace{1cm} $\square$

In fact, numerical evidence indicates that the function $F$ is increasing on $[-1, 1]$.

4. Lower bound for special weight functions

Using Yudin’s function in Theorem 2.2 gives an explicit lower bound for positive cubature formulae. The bound takes a particular simple form for the Chebyshev weight function.

**Theorem 4.1.** Every positive cubature formula for the Chebyshev weight function $(1 - ||x||^2)^{-1/2}$ on the ball $B^d$ whose nodes are inside $B^d$ satisfies
\[ N_s \geq \int_0^1 (1 - t^2)^{(d-2)/2} \, dt / \int_{\gamma_s}^1 (1 - t^2)^{(d-2)/2} \, dt, \] (4.1)

where \( \gamma_s \) is the largest zero of \( C_s^{(d+1)/2}(t) \).

**Proof.** We use \( F \) in the previous proposition, except that we take \( F \in \mathcal{F}_s(\lambda) \) instead of \( \mathcal{F}_{s-1}(\lambda) \), where \( \lambda = (d - 1)/2 \). For the Chebyshev weight, we use limit (2.3) to conclude that

\[
\min_{0 \leq r \leq 1} \sigma(F, r) = \min_{0 \leq r \leq 1} (F(1) + F(2r - 1))/2 = F(1)/2.
\]

We have \( \hat{F}_0 = f_0 g_0/h_0^2 \) and \( F(1) = \int_{\gamma_s}^1 f(u)(1 - u^2)^{s-1/2} \, du = f_0/h_0^2 \). Consequently,

\[
N_s \geq \sigma(F, 0)/\hat{F}_0 = F(1)/(2\hat{F}_0) = 1/(2g_0) = \frac{1}{2h_0^2 \int_{\gamma_s}^1 (1 - y^2)^{s-1/2} \, dy}
\]

and the estimate follows from the definition of \( h_0^2 \). \( \square \)

In the case of \( d = 1 \), we are looking at the quadrature formula for Chebyshev weight on the interval \([-1, 1]\). In this case, \( \gamma_s \) is the largest zero of the Chebyshev polynomial of the second kind \( C_s^1(t) = U_s(t) \). It follows that \( \gamma_s = \cos(\pi/(s + 1)) \).

Consequently, the bound in the theorem states that \( N_s \geq (s + 1)/2 \), which is sharp. The case \( d = 2 \) is stated in the following corollary.

**Corollary 4.2.** For the Chebyshev weight function on \( B^2 \),

\[
N_s \geq (j_1/2)s^2(1 + \mathcal{O}(s^{-1})) \geq 0.13622s^2(1 + \mathcal{O}(s^{-1})),
\]

where \( j_1 \) is the first positive zero of the Bessel function \( J_1(x) \).

**Proof.** We use the asymptotic formula for the largest zero \( \cos \theta_s \) of \( C_s^1 \) [1, p. 787] or [6, Corollary 2],

\[
\theta_s = \frac{j_2 - 1/2}{s} + \mathcal{O}(s^{-3}),
\]

where \( j_2 \) is the first positive zero of the Bessel function \( J_2(t) \). In the case of \( d = 2 \), the lower bound in the previous theorem takes the form \( N_s \geq 1/(1 - \gamma_s) \), where \( \gamma_s \) is the largest zero of \( C_s^{3/2}(t) \). It follows that

\[
N_s \geq \frac{1}{1 - \cos \theta_s} = \frac{2s^2}{j_1} \big(1 + \mathcal{O}(s^{-1})\big) = 0.136221 \ldots s^2(1 + \mathcal{O}(s^{-1}))
\]

for the Chebyshev weight function on \( B^2 \). \( \square \)

For \( s \) sufficiently large, this lower bound improves the lower bound given in (1.1) and (1.2), since those classical bounds give

\[
N_s \geq (s^2/8)(1 + \mathcal{O}(s^{-1})) = 0.125s^2(1 + \mathcal{O}(s^{-1})).
\]
In particular, this shows that the lower bound in (1.1) and (1.2) is far off for \( s \) large. The numerical computation of the bound in Corollary 4.2 shows that the bound in (4.1) starts to get better than that of (1.1) and (1.2) when \( s \geq 31 \). We give the numerical result of the first few improved bounds in the following table, in which we let

\[
N_s = \text{Bound in (1.1) and (1.2)} \quad \text{and} \quad N_s^* = \text{Bound in (4.1)}
\]

<table>
<thead>
<tr>
<th>( s )</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_s )</td>
<td>127</td>
<td>136</td>
<td>144</td>
<td>153</td>
<td>161</td>
<td>171</td>
<td>180</td>
</tr>
<tr>
<td>( N_s^* )</td>
<td>126.85</td>
<td>135.3</td>
<td>144.07</td>
<td>153.007</td>
<td>162.27</td>
<td>171.8</td>
<td>181.6</td>
</tr>
</tbody>
</table>

Let us point out that, in terms of asymptotics behavior in \( s \), the lower bounds given in Theorem 4.1 also work for weight functions \( W_m \) with \( m \) being an integer. Let us consider the case of \( d = 2 \) and \( m = 1 \). If \( \sum_{k=1}^{N} \lambda_k f(x_k), x_k \in B^d \), is a positive cubature formula of degree \( s \) for \( W_0(x) = 1/\sqrt{1 - \|x\|^2} \), then \( N \) satisfies the lower bound in Corollary 4.2. Setting \( f(x) = (1 - \|x\|) g(x) \) leads to a cubature formula \( \sum_{k=1}^{N} \lambda_k g(x_k) \) of degree \( s - 2 \) for \( W_1(x) = \sqrt{1 - \|x\|^2} \). Clearly, every positive cubature formula for \( W_1 \) can be derived this way, which shows that the number of nodes \( N_s \) for \( W_1 \) is bounded below by \( 0.13622(s + 2)^2(1 + O(s^{-1})) \), which is in the same order as \( 0.13622s^2(1 + O(s^{-1})) \) for \( s \) sufficiently large.

The lower bound for the Chebyshev weight function is obtained as the limit of the lower bound in Theorem 2.2 for \( \mu \to 0 \). Hence, we can expect improved lower bounds for the weight function \( W_\mu \) for \( \mu \) small. The most interesting case is naturally the case \( \mu = 1/2 \) for which the weight function \( W_\mu \) is a constant. In this case, \( \lambda = 1 \), and the graph of the function for small \( s \) indicates that \( F \) is an increasing function and that the lower bound in Corollary 2.3 takes the form

\[
N_s \geq \frac{1}{2} \int_{-1}^{1} F(t)(1 - t^2)^{-1/2} dt / \int_{-1}^{1} F(t)(1 - t^2)^{1/2} dt.
\] (4.2)

However, it is rather surprising that this lower bound appears to be weaker than the classical bound. We need an explicit formula for the function \( F \). Let \( \gamma_s = \cos \theta_s \). By Proposition 3.3, \( F(\cos \theta) = 0 \) for \( \theta \geq 2\theta_s \). Let \( f_\theta(x) = C_\theta^\circ(x) - C_\theta^\circ(\gamma_s) \) and define

\[
H_\theta(\theta, \phi, \xi) = (\cos(\theta - \phi) - \cos \xi)^{s-1}(\cos \xi - \cos(\theta + \phi))^{s-1}.
\]
Since \( f(\cos \xi) = 0 \) if \( \xi \geq \theta_s \), it follows that for \( 0 \leq \theta \leq \theta_s \),
\[
F(\cos \theta) = \frac{c_\lambda}{(\sin \theta)^{2\lambda - 1}} \left[ \int_{0}^{\theta_1 - \theta} \int_{\theta - \phi}^{\theta + \phi} f_\lambda(\cos \xi) H_\lambda(\theta, \phi, \xi) \sin \xi \, d\xi \, \sin \phi \, d\phi \right. \\
+ \left. \int_{\theta - \phi}^{\theta_1} \int_{\theta - \phi}^{\theta_1} f_\lambda(\cos \xi) H_\lambda(\theta, \phi, \xi) \sin \xi \, d\xi \, \sin \phi \, d\phi \right]
\]
and for \( \theta_s \leq \theta \leq 2\theta_s \)
\[
F(\cos \theta) = \frac{c_\lambda}{(\sin \theta)^{2\lambda - 1}} \int_{\theta - \phi}^{\theta_1} \int_{\theta - \phi}^{\theta_1} f_\lambda(\cos \xi) H_\lambda(\theta, \phi, \xi) \sin \xi \, d\xi \, \sin \phi \, d\phi.
\]

For \( \mu = \frac{1}{2} \) we have \( \lambda = 1 \) and \( H_1(\theta, \phi, \xi) = 1 \), and \( C_1^1 \) is the Chebyshev polynomial of the second kind, so that we can derive an explicit formula of \( F \). The formula is still complicated, but it allows us to draw the graph of \( F \) which shows that it is indeed increasing and compute the lower bound in \((4.2)\) using a computer. The numerical computation for \( s \) up to 200, however, shows that this bound is much weaker than the classical bound in \((1.1)\) and \((1.2)\) (for \( n = 200 \), it is 2697 vs. 5151). It appears that Yudin’s function fails to provide a better bound for the case of \( \mu = \frac{1}{2} \). An interesting question is to construct another family of increasing function \( F \) that will maximize the lower bound in \((4.2)\).

References